A specialisation of the Bump-Friedberg L-function

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Abstract

We study the restriction of the Bump-Friedberg integrals to the complexe line $\{(s, 2s), s \in \mathbb{C}\}$. It has a simple theory, very close to that of the Asai L-function. It is an integral representation of the product $L(s,\pi)L(2s,\Lambda^2,\pi)$ which we denote by $L^{lin}(s,\pi)$, when π is a cuspidal automorphic representation of $GL(k,A)$ for A the adeles of a number field. In the global case, when k is even, we show that for a cuspidal automorphic representation π, the partial L-function $L^{lin,S}(s,\pi)$ has a pole at $1/2$, if and only if π admits a global period, this gives another proof of a theorem of Jacquet and Friedberg, asserting that π has a global period if and only if $L(1/2,\pi) \neq 0$ and $L(1,\Lambda^2,\pi) = \infty$. We also show functional equations in the global and local cases (except at the archimedean places). In the local non-archimedean case, we show that linear periods of π are related to exceptional poles of $L^{lin}(s,\pi)$ when π is unitary, this property should hold in general. We also deal with the case k odd, then the partial L-function is holmorphic in a neighbourhood of $Re(s) \geq 1/2$.

Introduction

In this paper, we study the retriction of the integrals of two complex variables (s_1, s_2) defined in [B-F], and attached to global and local smooth complex representations of $GL(2n)$, to the line $s_2 = 2s_1$.
It turns out that this restriction has a theory very close to the theory of Asai L-functions, whose Rankin-Selberg theory, initiated by Flicker, is quite complete now (see [F], [F-Z], [K], [AKT], [AR], [M1], [M2], [M3]). Hence, though some of the results obtained in the sequel are special cases of the results of [B-F], in particular the Euler factorisation, the global functional equation and the unramified computation, we reprove them because we feel that this L function obtained by restriction, somehow deserves its own theory. Another reason is that we started the writing of this paper without knowing the results of [B-F].

The paper is organised as follows. In the first section, we introduce the materiel needed.

In the first paragraphs of the second section, we first define the L function $L^{lin}(s,\pi)$ for a generic representation π of $GL(2n,F)$ (Theorem 2.1), when F is a non-archimedean local field. We prove the functional equation (Theorem 2.2). We then relate the occurence of linear periods for π and exceptional poles of $L^{lin}(s,\pi)$ when π is unitary (Theorem 2.4). To end the non-archimedean theory, we compute the Rankin-Selberg integrals when π is unramified in Paragraph 2.4.
We don't do much in the archimedean case (Paragraph 2.5), however we prove results of convergence and non-vanishing of the archimedean integrals, that we use in the global situation.

Section 3 is devoted to the global theory. We take π a cuspidal automorphic representation of $GL(2n,A)$, for A the adele ring of a number field k. We first study the integrals $I(s,\phi,\Phi)$ associated to a cusp form in the space of π and a Schwartz function Φ on A^n, via the theory of mirabolic

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Eisenstein series, to obtain their meromorphicity, functional equation as well as the location of their possible poles in Theorem 3.1. Then we prove the equality of these integrals (Theorem 3.2) with the Rankin-Selberg integrals $\Psi(s, W_\phi, \Phi)$ obtained by integrating the Whittaker functions associated to ϕ, and thus get the Euler factorisation in Paragraph 3.2.
In the last part, we define the partial L-function $L^{lin,S}(s,\pi)$, and show that it is meromorphic, holomorphic for $Re(s) > 1/2$, and that it has a pole at $1/2$ if and only if π has a global period (Theorem 3.3). We deduce from this the theorem of Friedberg and Jacquet discussed in the abstract (Theorem 3.4).

In Section 4, we give the results for the odd case. The global Rankin-Selberg integrals are holomorphic this time, and we prove that the partial L-function is holomorphic in a neighbourhood of $Re(s) \geq 1/2$.

Finally, in Section 5, we state a conjecture about the classification of generic representations of $GL(k,F)$ distinguished by a maximal Levi, for F non-archimedean, in terms of discrete series, which is related with the multiplicativity relation of the local L function.

1 Preliminaries

1.1 The groups involved

In this work, k will be a number field, A will be its adele ring.
The letter F will denote a local field. If F is $\mathbb{R}$ or $\mathbb{C}$, we will denote by $|.|$ the standard absolute value on $\mathbb{R}$, and the square of the modulus on $\mathbb{C}$. If F is non-archimedean, we denote by $\mathfrak{O}$ its ring of integers, $\mathfrak{P} = \varpi\mathfrak{O}$ the maximal ideal of this ring, for ϖ a uniformiser of F. We will denote by q the cardinality of $\mathfrak{O}/\mathfrak{P}$, and by $|.|$ the normalised absolute value on F such that $|\varpi| = q^{-1}$. We will also denote by $|.|$ the absolute value of A^* obtained as the product of the normalised absolute values for all places. In every case, if χ is a character of F, we denote by $Re(\chi)$ the real r such that $|\chi(t)| = |t|^r$.
We denote $GL(n,.)$ by G_n for $n \geq 1$, and we set $G_0 = \{1\}$. We denote by A_n the torus of diagonal matrices in G_n. We will sometimes denote by $|.|$ the positive character $|\ | \circ det$ of G_n over A or F. The standard maximal Levi subroup of type (p,q) where $p+q=n$ of G_n will be denoted by $M_{p,q}$, and N_n will be the unipotent radical of the standard Borel subgroup of G_n. We will denote by Z_n the center of G_n.
For $n \geq 2$ we denote by U_n the group of matrices of the form $\begin{pmatrix} I_{n-1} & V \\ & 1 \end{pmatrix}$, and by U_n^- its opposite.
For R a ring, and for x in R^{n-1}, we denote by $u(x)$ the matrix $\begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}$ of $U_n(R)$, and by $u^-(x)$ the matrix $\begin{pmatrix} 1 & \\ x & I_{n-1} \end{pmatrix}$ of $U_n^-(R)$. We denote by P_n the mirabolic subgroup $G_{n-1}U_n$ of G_n.
If g belongs to G_n, we denote by $L_n(g)$ the last row of g.

We denote by w_{2n} the element of the symetric group $\mathfrak{S}_{2n}$ defined by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n-1 & 2n \\ 1 & 3 & \dots & 2n-3 & 2n-1 & 2 & 4 & \dots & 2n-2 & 2n \end{pmatrix}$$

and by w_{2n+1} the element of the symetric group $\mathfrak{S}_{2n+1}$ defined by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n & 2n+1 \\ 1 & 3 & \dots & 2n-3 & 2n-1 & 2n+1 & 2 & \dots & 2n-2 & 2n \end{pmatrix}.$$

We will also need the matrix w'_{2n+1} of G_{2n+1} corresponding to the permutation:

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & n+3 & \dots & 2n+1 \\ 2 & 4 & \dots & 2n-2 & 2n & 2n+1 & 1 & 3 & \dots & 2n-1 \end{pmatrix}$$

as well as the matrix w'_{2n} of G_{2n} corresponding to the permutation:

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & n+3 & \dots & 2n \\ 1 & 3 & \dots & 2n-3 & 2n-1 & 2n & 2 & 4 & \dots & 2n-2 \end{pmatrix}$$

We will often write $w(h) = whw^{-1}$ for h in G_l and w a matrix corresponding to an element of $\mathfrak{S}_l$.

We denote by $M_{n,n}$ the standard Levi subgroup of G_{2n} associated with the partition (n,n), and by $M_{n,n-1}$ the standard Levi subgroup of G_{2n-1} associated with the partition $(n,n-1)$. We then denote by $H_{n,n}$ the group $w_{2n}(M_{n,n})$ and by $H_{n,n-1}$ the group $w_{2n-1}(M_{n,n})$.
Moreover, if $h = w_{2n}(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix})$ belongs to $H_{n,n}$, we will sometimes write $h = h(h_1,h_2)$. We will also write $h = h(h_1,h_2)$ for $h = w_{2n-1}(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix})$ in $H_{n,n-1}$.If C is a subset of G_{2n}, we write C^σ for $C \cap H_{n,n}$. If C is a subset of G_{2n-1}, we write C^σ for $C \cap H_{n,n-1}$.

1.2 Smooth functions and representations

When G is the points of an algebraic group defined over $\mathbb{Z}$ on F or A, we denote by $Sm(G)$ the category of smooth complex G-modules. Every representation we will consider from now on will be smooth and complex.
We will denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on H, and int is the action given by $(int(n)f)(h) = f(n^{-1}hn)$, of $N_G(H)$ smooth functions f with compact support on H, then $\mu \circ int(n) = \delta_H^{-1}(n)\mu$ for n in $N_G(H)$.

Now G is locally compact totally disconnected. If (π, V) belongs to $Sm(G)$, H is a closed subgroup of G, and χ is a character of H, we denote by $V(H,\chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V. This space is actually stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G, which fixes χ.
The space $V(H,\chi)$ is $N_G(\chi)$-stable. Thus, if L is a closed-subgroup of $N_G(\chi)$, and μ is a (smooth) character of L, the quotient $V_{H,\chi} = V/V(H,\chi)$ (that we simply denote by V_H when χ is trivial) becomes a smooth L-module for the action $l.(v + V(H,\chi)) = \mu(l)\pi(l)v + V(H,\chi)$ of L on $V_{H,\chi}$.
We say that a representation V of G is (H,χ)-distinguished (we will sometimes omit H when the context is clear, and χ when χ is trivial) if $V_{H,\chi}$ is not reduced to zero, or equivalently when $Hom_H(\pi,\chi) \neq \{0\}$.
If H is a closed subgroup of a p-adic group G, and (ρ, W) belongs to $Sm(H)$, we define the objects $(ind_H^G(\rho), V_c = ind_H^G(W))$ and $(Ind_H^G(\rho), V = Ind_H^G(W))$ of $Sm(G)$ as follows. The space V is the space of smooth functions from G to W, fixed under right translation by the elements of a compact open subgroup U_f of G, and satisfying $f(hg) = \rho(h)f(g)$ for all h in H and g in G. The space V_c is the subspace of V, consisting of functions with support compact mod H, in both cases, the action of G is by right translation on the functions. We will sometimes denote V by $\mathcal{C}^\infty(H\backslash G, \rho)$, and V_c by $\mathcal{C}_c^\infty(H\backslash G, \rho)$.

If $G = G_{2n}(A)$, $H = H_{n,n}(A)$, and π is a cuspidal representation of G, we say that π has an (H,χ)-period if there is a cusp form in the space of π such that

$$\int_{Z_{2n}(A)H_{n,n}(k)\backslash H_{n,n}(A)} \phi(h)\chi^{-1}(h)dh$$

is nonzero.

We will denote by $\mathcal{S}(F^n)$ the Schwartz space of functions (smooth and rapidly decreasing) on F^n when F is archimedean, and by $\mathcal{C}_c^\infty(F^n)$ the Schwartz space of smooth functions with compact support on F^n when F is non-archimedean. We denote by $\mathcal{S}(A^n)$ the space of Schwartz functions on A^n, which is by definition the space of linear combinations of decomposable functions $\Phi = \prod_\nu \Phi_\nu$, with Φ_ν in $\mathcal{S}(k_\nu^n)$ when ν is an archimedean place, and in $\mathcal{C}_c^\infty(k_\nu^n)$ when ν is non-archimedean. On these spaces, there is a natural action of either $G_n(F)$, or $G_n(A)$. In every case, if θ is a nontrivial character of F or A, we will denote by $\widehat{\Phi}^\theta$ or $\widehat{\Phi}$ the Fourier transform of a Schwartz function Φ, with respect to θ-self-dual Haar measure.

1.3 Reminder about derivatives

In this paragraph, as well as in the next section, if G is an algebraic group defined over F, we will write by abuse of notation G for its F-points.

For the rest of this paragraph, F is non archimedean, we recall facts from [B-Z]. We define the following functors:

- The functor Φ^- from $Sm(P_k)$ to $Sm(P_{k-1})$ such that, if (π, V) is a smooth P_k-module, $\Phi^- V = V_{U_k,\theta}$, and P_{k-1} acts on $\Phi^-(V)$ by $\Phi^-\pi(p)(v + V(U_k,\theta)) = \delta_{U_k}(p)^{-1/2}\pi(p)(v + V(U_k,\theta))$.
- The functor Φ^+ from $Sm(P_{k-1})$ to $Sm(P_k)$ such that, for π in $Sm(P_{k-1})$, one has $\Phi^+\pi = ind_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta)$.
- The functor Ψ^- from $Sm(P_k)$ to $Sm(G_{k-1})$, such that if (π, V) is a smooth P_k-module, $\Psi^- V = V_{U_k,1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^-\pi(g)(v) + V(U_k,1) = \delta_{U_k}(g)^{-1/2}\pi(p)(v + V(U_k,1))$.
- The functor Ψ^+ from $Sm(G_{k-1})$ to $Sm(P_k)$, such that for π in $Sm(G_{k-1})$, one has $\Psi^+\pi = ind_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes 1) = \delta_{U_k}^{1/2}\pi \otimes 1$.

The functors Φ^-, Φ^+, Ψ^-, and Ψ^+ are exact. Every representation τ of P_n has a natural filtration $0 \subset \tau_{n-1} \subset \cdots \subset \tau_0 = \tau$, where $\tau_k = {\Phi^+}^k {\Phi^-}^k \tau$. Moreover, writing $\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau)$ the representation of G_{n-k}, one has $\tau_k/\tau_{k+1} = (\Phi^+)^k \Psi^+ \tau^{(k+1)}$. If τ has finite length, the same holds for every $\tau^{(k)}$.

Recalling that we call a generic representation of G_n an irreducible representation π which is (N_n, θ)-distinguished, genericity is characterised in terms of derivatives by the fact that $\pi^{(n)} = \mathbf{1}$.

2 The local theory

We start with the non-archimedean case, we get a quite complete theory, with definition of the local L-functions, functional equations, unramified computation, and relation between distinction and exceptional poles of L-functions.

2.1 Definition of the local non-archimedean L-function

Let θ be a nonzero character of F. Let π be generic representations of G_{2n}, W belong to the Whittaker model $W(\pi, \theta)$, and Φ be a function in $C_c^\infty(F^n)$. We denote by χ a character of $H_{n,n}$ of the form $h(h_1, h_2) \mapsto \alpha(det(h_1)/det(h_2))$, for α a character of F^*, and by δ the character

$h(h_1,h_2) \mapsto |h_1|/|h_2|$ of $H_{n,n}(F)$.
We define formally the integral

$$\Psi(s,W,\chi,\Phi) = \int_{N_{2n}\cap H_{n,n}\backslash H_{n,n}} W(h)\Phi(L_n(h_2))|h|^s\chi(h)\delta(h)^{-1/2}dh.$$

This integral is convergent for $Re(s)$ large, and defines an element of $\mathbb{C}(q^{-s})$:

Theorem 2.1. *There is a real number $r_{\pi,\chi}$, such that each integral $\Psi(s,W,\chi,\Phi)$ converges for $Re(s) > r_\pi$. Moreover, when W and Φ vary in $W(\pi,\theta)$ and $C_c^\infty(F^n)$ respectively, they span a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$ in $\mathbb{C}(q^{-s})$, generated by an Euler factor which we denote $L^{lin}(s,\pi,\chi)$.*

Proof. The convergence for $Re(s)$ greater than a real $r_{\pi,\chi}$ is classical, it is a consequence of the asymptotic expansion of the restriction of W to the torus A_n, which can be found in [J-P-S2] for example. The fact that these integrals span a fractinal ideal of $\mathbb{C}[q^s,q^{-s}]$ is a consequence of the observation that $\Psi(s,W,\chi,\Phi)$ is multiplied by $|h|^{-s}\chi^{-1}(h)\delta^{1/2}(h)$ when one replaces W and Φ by their right translate under h.
Denoting by c_π the central character of π, and by $K_{n,n}$ the points of $H_{n,n}$ on $\mathfrak{O}$, we write, thanks to Iwasawa decomposition, the integral $\Psi(s,W,\chi,\Phi)$ as

$$\int_{K_{n,n}}\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma} W(pk)|p|^{s-1/2}\chi(pk)\left(\int_{F^*}\Phi(aL_n(k))c_\pi(a)|a|^{2ns}d^*a\right)dpdk$$

As in [J-P-S2], for any ϕ in $\mathcal{C}_c^\infty(N_{2n}\backslash P_{2n},\theta)$, there is a W such that $W_{|P_{2n}}$ as ϕ. Such a ϕ in right invariant under an open subgroup U of $K_{n,n}$, which also fixes χ. We then chose Φ to be the characteristic function of $\{L_n(h_2), h_2 \in U\}$, the integral then reduces to a positive multiple of

$$\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma}\phi(p)|p|^{s-1/2}\chi(p)dp,$$

we now see that for ϕ well-chosen, this last integral is 1, i.e. $\Psi(s,W,\chi,\Phi)$ is 1. This implies that the generator of the fractional ideal spanned by the $\Psi(s,W,\chi,\Phi)$ can be chosen as an Euler factor.

□

2.2 The local functional equation

Now we want to prove that these Rankin-Selberg integrals admit a functional equation.
As usual this will be a consequence of a multiplicity one theorem for invariant bilinear maps on $W(\pi,\theta)\times C_c^\infty(F^n)$.

We will need the following result from [M4]. Let δ' be the character of $H_{n,n-1}$ defined by $h(h_1,h_2)\mapsto |h_1|/|h_2|$, then we have the following isomorphisms.

Proposition 2.1. *For an irreducible representation ρ of P_{2n-1}, and a character μ of $P_{2n-1}\cap H_{n,n}$. One has*

$$Hom_{P_{2n}\cap H_{n,n}}(\Phi^+(\rho),\mu) \simeq Hom_{P_{2n-1}\cap H_{n,n-1}}(\rho,\mu\delta^{-1/2}).$$

For an irreducible representation ρ of P_{2n-2}, and a character μ of $P_{2n-1}\cap H_{n,n}$, one has

$$Hom_{P_{2n-1}\cap H_{n,n-1}}(\Phi^+(\rho),\mu) \simeq Hom_{P_{2n-2}\cap H_{n-1,n-1}}(\rho,\mu\delta'^{1/2}).$$

Using the Bernstein-Zelevinsky filtration of the retriction to P_{2n} of π, the previous proposition implies the following result.

Proposition 2.2. *Let θ be a positiveon trivial character of F, and π be a generic representation of G_{2n}, then $Hom_{P_{2n}^\sigma}(\pi,\chi|.|^{-s})$ has dimension 1 for every value of q^{-s}, except a finite number.*

Proof. Indeed, the restriction of π to P_{2n} has a finite composition series, in which every irreducible subquotient is of the form $(\Phi^+)^{2n-k-1}\Psi^+(\rho)$, for $k \leq 2n-1$, and ρ an irreducible representation of G_k. But a nonzero element of $Hom_{P_{2n}^\sigma}(\pi, \chi|.|^{-s})$ must induce a nonzero element of $Hom_{P_{2n}^\sigma}((\Phi^+)^{2n-k-1}\Psi^+(\rho), \chi|.|^{-s})$ for some k and ρ. If $k \geq 1$, applying repeatedly Proposition 2.1, and considering the central character of ρ, we see that this latter space is zero except for a finite number of values of q^{-s}. For $k = 0$, then ρ is $\mathbf{1}$, and Proposition 2.1 implies that $Hom_{P_{2n}^\sigma}((\Phi^+)^{2n-1}\Psi^+(\mathbf{1}), \chi|.|^{-s})$ is of dimension 1. The result follows. $\square$

From this we deduce the following.

Proposition 2.3. *For almost every s, one has $dim_{\mathbb{C}}[Hom_{G_{2n}^\sigma}(\pi \otimes C_c^\infty(F^n), \chi^{-1}\delta^{1/2}|.|^{-s})] \leq 1$.*

Proof. Because of π's central character, the space $Hom_{H_{n,n}}(\pi, |.|^{-s}\chi^{-1}\delta^{1/2})$ is zero except for a finite number of values of q^{-s}, hence the space $Hom_{H_{n,n}}(\pi \otimes \mathcal{C}_c^\infty(F^n), \chi^{-1}\delta^{1/2}|.|^{-s})$ is equal to $Hom_{H_{n,n}}(\pi \otimes \mathcal{C}_{c,0}^\infty(F^n), \chi^{-1}\delta^{1/2}|.|^{-s})$ except for those values ($\mathcal{C}_{c,0}^\infty(F^n)$ being the subspace of $\mathcal{C}_c^\infty(F^n)$ consisting of functions vanishing at zero). But

$$Hom_{H_{n,n}}(\pi \otimes \mathcal{C}_{c,0}^\infty(F^n), \chi^{-1}\delta^{1/2}|.|^{-s}) \simeq Hom_{H_{n,n}}(\pi \otimes ind_{H_{n,n}\cap P_{2n}}^{H_{n,n}}(1), \chi^{-1}\delta^{1/2}|.|^{-s})$$

$$\simeq Hom_{H_{n,n}}(\pi, Ind_{H_{n,n}\cap P_{n,n}}^{H_{n,n}}(|.|^{-s+1/2}\chi^{-1})) \simeq Hom_{H_{n,n}\cap P_{2n}}(\pi, |.|^{-s+1/2}\chi^{-1}),$$

the first isomorhism identifying $H_{n,n} \cap P_{2n}\backslash H_{n,n}$ with $F^n - \{0\}$, and the last by Frobenius reciprocity law. Now the result follows from Proposition 2.2. $\square$

Then we obtain as a consequence the following functional equation:

Theorem 2.2.

$$\Psi(1/2-s, \tilde{W}, \chi^{-1}\delta^{1/2}, \widehat{\Phi}^\theta)/L^{lin}(1/2-s, \pi^\vee, \chi^{-1}\delta^{1/2}) = \epsilon^{lin}(s, \pi, \chi, \theta)\Psi(s, W, \chi, \Phi)/L^{lin}(s, \pi, \chi)$$

for $\epsilon^{lin}(s, \pi, \chi, \theta)$ a unit of $\mathbb{C}[q^s, q^{-s}]$, where $\tilde{W} : g \mapsto W(w{}^tg^{-1})$ with w is the permutation matrix corresponding to $i \mapsto 2n - i$.

2.3 Exceptional poles and distinction

We now talk about the link between the exceptional poles of $L^{lin}(s, \pi, \chi)$ and linear periods. The most complete result we get is for unitary (generic) representations, but we intend to remove this restriction in the future. Almost everything in this paragraph is a straightforward adaptation from results of [M1], so many proofs will be very sketchy, or not even written. We will give the results in terms of the function $L^{lin}(s, \pi, \chi^{-1}\delta^{1/2})$ rather than in terms of the function $L^{lin}(s, \pi, \chi)$. Of course one can then translate the results in terms of $L^{lin}(s, \pi, \chi)$.

Definition 2.1. *We say that a pole s_0 of order k of $L^{lin}(s, \pi, \chi^{-1}\delta^{1/2})$ is exceptional, if it cannot occur with this order in any integral $\Psi(s, W, \chi^{-1}\delta^{1/2}, \Phi)$ when Φ vanishes at zero.*

We give the following characterizations of an exceptional pole, which are useful.

Proposition 2.4. *Let s_0 be a pole of order k of $L^{lin}(s, \pi, \chi^{-1}\delta^{1/2})$, then for W in $W(\pi, \theta)$ and Φ in $\mathcal{C}^\infty(F^n)$, the Laurent expansion of $\Psi(s, W, \chi\delta^{1/2}, \Phi)$ at s_0 is of the form $B_{s_0}(W, \Phi)/(1 - q^{s_0 - s}) + \ldots.$. Then s_0 is exceptional if and only if the bilinear form B_{s_0} factorises as $B_{s_0}(W, \phi) = \Lambda_{s_0}(W)\phi(0)$, for some nonzero linear form L_{s_0}, which is $|.|^{-s_0}\chi$-invariant under $H_{n,n}$.*
One also sees that s_0 is exceptional if and only if it occurs with order strictly smaller than k in every integral of the form $\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma} W(p)\chi^{-1}(p)\delta^{1/2}(p)|p|^{s-1/2}dp$.

It is worth introducing another Euler factor, related to exceptional poles.

Theorem 2.3. *There is a real number r_π, such that the integrals*

$$\Psi_{(0)}(s, W, \chi) = \int_{N_{2n}^\sigma \backslash P_{2n}^\sigma} W(p)\chi^{-1}(p)|p|^{s-1/2}dp,$$

for W in $W(\pi,\theta)$ are absolutely convergent for $Re(s) > r_\pi$. They generate a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$, which is the same as the one generated by the integrals $\Psi(W, \Phi, \chi^{-1}, s)$, with Φ vanishing at zero, this fractional ideal contains 1. In particular it is generated by a unique Euler factor that we denote by $L^{lin}_{(0)}(s, \pi, \chi)$. Moreover

$$L^{lin}(s, \pi, \chi^{-1}\delta^{1/2})/L^{lin}_{(0)}(s, \pi, \chi^{-1}\delta^{1/2})$$

has simple poles, which are exactly the exceptional poles of $L^{lin}(s, \pi, \chi^{-1}\delta^{1/2})$.

In particular, this allows us to define, for any generic representation π, a non zero linear form on its space, which is $\chi|.|^{1/2-s}$-invariant under P^σ_{2n}.

Proposition 2.5. *Let π be a generic representation of G_{2n}, then*

$$\Lambda_{\pi,\chi^{-1},s} : W \mapsto \Psi_{(0)}(s, W, \chi^{-1})/L^{lin}_{(0)}(s, \pi, \chi^{-1})$$

is a nonzero element of $Hom_{P^\sigma_{2n}}(\pi, \chi|.|^{1/2-s})$. The linear form

$$\Lambda'_{\pi,\chi^{-1},s} : W \mapsto \Psi_{(0)}(s, W, \chi^{-1})/L^{lin}(\pi, \chi^{-1}, s)$$

is a possibly null element of $Hom_{P^\sigma_{2n}}(\pi, \chi|.|^{1/2-s})$.

We now state a consequence of Bernstein's characterization of unitary representations via exponents of derivatives. For the next Lemma, we consider χ of the form $\chi(h(h_1, h_2)) = \alpha(det(h_1)/det(h_2))$ for α a character of F^*, which satisfies $Re(\alpha) \leq 0$.

Lemma 2.1. *Let χ be as discussed just above, and π be a generic unitary representation of G_{2n}. Then $Hom_{P^\sigma_{2n}}(\pi, \chi)$ is of dimension 1.*

Proof. According to the criterion 7.4 of [B], the central characters of every irreducible subquotient of the nonzero derivatives $\pi^{(k)}$ are of real part $> (k - 2n)/2$, except the central character of the highest derivative $\pi^{(2n)} = \mathbf{1}$. Now the restriction of (π, V) to P_{2n} has a filtration $\{0\} = V_n \subset \cdots \subset V_0 = V$, such that V_{k-1}/V_k is $(\Psi^+)^{k-1}\Phi^+(\pi^{(k)})$, so a nonzero element of $Hom_{P^\sigma_{2n}}(\pi, \chi)$ must induce a nonzero element of $Hom_{P^\sigma_{2n}}((\Phi^+)^{k-1}\Psi^+(\rho), \chi)$ for some k, and an irreducible subquotient ρ of $\pi^{(k)}$. We claim that all these spaces are zero, except $Hom_{P^\sigma_{2n}}((\Phi^+)^{2n-1}\Psi^+(\mathbf{1}), \chi)$, which is of dimension 1.
Indeed,

$$Hom_{P^\sigma_{2n}}((\Phi^+)^{k-1}\Psi^+(\rho), \chi) \simeq Hom_{P^\sigma_{2n+1-k}}(\Psi^+(\rho), \chi\mu_k) \simeq Hom_{G^\sigma_{2n-k}}(\rho, \chi\mu_k|.|^{-1/2}),$$

where $\mu_k = \delta^{-1/2}$ if k is even, and 1 if k is odd. Let k be $< 2n$, denoting by $z(t)$ the matrix tI_{2n-k}, one has

$$Re(\chi(z(t))\mu_k(z(t))|z(t)|^{-1/2}) = Re(\chi(z(t))\mu_k(z(t)) + (k - n)/2,$$

and $Re(\chi(z(t))\mu_k(z(t)) \leq 0$, hence $Re(\chi(z(t))\mu_k(z(t))|z(t)|^{-1/2}) \leq (k-n)/2$, whereas $Re(c_\rho(t)) > (k-n)/2$, and this implies $Hom_{P^\sigma_{2n}}((\Phi^+)^{k-1}\Psi^+(\rho), \chi) = \{0\}$. Finally $Hom_{P^\sigma_{2n}}((\Phi^+)^{2n-1}\Psi^+(\mathbf{1}), \chi)$ is of dimension 1. Hence $Hom_{P^\sigma_{2n}}(\pi, 1)$ is of dimension ≤ 1, and actually of dimension exactly 1 according to Proposition 2.5. ∎

This lemma implies the following theorem:

Theorem 2.4. *Let π be a generic unitary representation of $G_{2n}(F)$, and χ a character of $H_{n,n}$ of the form $\alpha(det(h_1)/det(h_2))$, with $Re(\alpha) \geq 0$ (in particular χ can be $\mathbf{1}$) then π is χ-distinguished if and only if $L^{lin}(s,\pi,\chi^{-1}\delta^{1/2})$ has an exceptional pole at 0.*

Proof. The fact that 0 being an exceptional pole of $L^{lin}(s,\pi,\chi^{-1}\delta^{1/2})$ implies that π is χ-distinguished has already been noticed, and is almost by definition. Conversely, if π is χ-distinguished, according to Proposition 2.1, we have $dim[Hom_{P_{2n}^\sigma}(\pi^\vee,\chi^{-1})] = 1$. But as π is χ-distinguished, then $\pi^\vee$ is χ^{-1}-distinguished (because $\pi^\vee \simeq g \mapsto \pi({}^tg^{-1})$), and as it is a result from [J-R], that

$$dim[Hom_{G_{2n}^\sigma}(\pi^\vee,\chi^{-1})] = 1,$$

we deduce that $Hom_{G_{2n}^\sigma}(\pi^\vee,\chi^{-1}) = Hom_{P_{2n}^\sigma}(\pi^\vee,\chi^{-1})$.
As in Theorem 2.1 of [M1], this implies:

$$\Psi(1/2,\tilde{W},\chi,\widehat{\Phi})/L^{lin}(1/2,\pi^\vee,\chi) = c\Lambda'_{1/2,\pi^\vee,\chi^{-1}}(\tilde{W})\Phi(0)$$

for a positive constant c. The functional equation then shows that

$$\Psi(0,W,\chi^{-1}\delta^{1/2}\Phi)/L^{lin}(0,\pi,\chi^{-1}\delta^{1/2}) = 0$$

whenever $\Phi(0) = 0$. But we already saw that one can choose W and Φ vanishing at zero, such that $\Psi(s,W,\chi^{-1}\delta^{1/2},\Phi)$ is 1, hence 0 is a pole of $L^{lin}(0,\pi,\chi^{-1}\delta^{1/2})$, which must be exceptional as the quotient $\Psi(0,W,\chi^{-1}\delta^{1/2}\Phi)/L^{lin}(0,\pi,\chi^{-1}\delta^{1/2})$ vanishes whenever $\Phi(0) = 0$. ☐

Remark 2.1. It is easy to see, as a consequence of the fact that the restriction of Whittaker functions in $W(\pi,\theta)$ to P_{2n} have compact support modulo center when π is cuspidal, that every pole of $L^{lin}_{(0)}(s,\pi,\delta^{1/2}) = 1$, hence $L^{lin}(s,\pi,\delta^{1/2})$ is equal to

$$\prod_{\{s_0,\pi\ is\ |.|^{-s_0}-distinguished\}} 1/(1-q^{s_0-s}).$$

2.4 The unramified computation

Here we show that the local Rankin-Selberg integrals give the expected L-function at the unramified places, we take $\chi = \mathbf{1}$ here.
Let π^0 be an unramified generic representation of $GL(2n,F)$, and W^0 the normalised spherical Whittaker function in $W(\pi^0,\theta)$ (here θ has conductor $\mathfrak{O}$), and let Φ^0 the characteristic function of $\mathfrak{O}^n$. We will use the notations of section 3 of [F]. We recall that π^0 is a commuting product (in the sense of [B-Z], i.e. corresponding to normalised parabolic induction) $\chi_1 \times \cdots \times \chi_{2n}$ of unramified characters, and we denote $\chi_i(\varpi)$ by z_i. Then it is well-know that if λ is an element of $\mathbb{Z}^{2n}$, then $W(\varpi^\lambda)$ is zero unless λ belongs to the set Λ^+ consisting of λ's satisfying $\lambda_1 \geq \cdots \geq \lambda_{2n}$, in which case $W(\varpi^\lambda) = \delta_{B_{2n}}^{1/2}(\varpi^\lambda)s_\lambda(z)$, where $s_\lambda(z) = det(z_i^{\lambda_j+n-j})/det(z_i^{n-j})$.
In this case, using Iwasawa decomposition, denoting by Λ^{++} the subset of Λ^+ with $\lambda_{2n} \geq 0$, and writing a' for $(a_1,a_3,\dots,a_{2n-1})$ and a'' for $(a_2,a_4,\dots,a_{2n})$, one has the identities

$$\Psi(s,W^0,\Phi^0) = \int_{A_{2n}} W^0(a)\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\Phi^0(a_{2n})|a|^s\delta(a)^{-1/2}da =$$

$$\int_{A_{2n}} W^0(a)\Phi^0(a_{2n})\delta_{B_{2n}}^{-1/2}(a)|a|^s da = \sum_{\lambda\in\Lambda^{++}} s_\lambda(z)q^{-s.tr\lambda} = \sum_{\lambda\in\Lambda^{++}} s_\lambda(q^{-s}z)$$

But this las sum is, as noticed in [F], according to combinatorial identites by Macdonald, to:

$$\prod_i(1-z_iq^{-s})\prod_{j<k}(1-z_jz_kq^{-2s}) = L(\pi^0,s)L(\pi^0,\Lambda^2,2s).$$

We end with the archimedean theory. The results we get are sufficient for the aim of this paper, however the theory is very incomplete, we will hopefully come back at it later.

2.5 Convergence and non vanishing of the archimedean integrals

Here F is archimedean, θ is a unitary character of F, and π is a generic unitary representation of G_{2n}, as in Section 2 of [J-S], to which we refer concerning this vocabulary. We denote by $W(\pi,\theta)$ its smooth Whittaker model.
We denote by δ the character $h(h_1,h_2)\mapsto |h_1|/|h_2|$ of $H_{n,n}(F)$.

We now define formally the following integral, for W in $W(\pi,\theta)$, and Φ in $\mathcal{S}(F^n)$:

$$\Psi(s,W,\Phi)=\int_{N_{2n}\cap H_{n,n}\backslash H_{n,n}} W(h)\Phi(L_n(h_2))|h|^s\delta(h)^{-1/2}dh.$$

We first state a proposition concerning the convergence of this integral:

Proposition 2.6. *The integral $\Psi(s,W,\Phi)$ is absolutely convergent for $s\geq 1/2-\epsilon$, for some positive real ϵ. In particular it defines a holomorphic function on this half plane.*

Proof. It is a consequence of Iwasawa decomposition, that to prove this statement, it is enough to prove it for the integral

$$\int_{A_{2n-1}} W(a)|a|^s\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\delta(a)^{-1/2}d^*a,$$

where $a'=(a_1,\dots,a_{2n-1})$, and $a''=(a_2,\dots,a_{2n-2},1)$. However $\delta_{B_n}^{-1}(a')\delta_{B_n}^{-1}(a'')\delta(a)^{-1/2}=\delta_{B_{2n}}^{-1/2}(a)=\delta_{B_{2n-1}}^{-1/2}(a)|a|^{-1/2}$. But according to Section 4 of [J-S2], writing t_i for a_i/a_{i+1} there is a finite set X consisting of functions which are products of polynomials in the logarithm of the $|t_i|$'s and a character $\chi(a)=\prod_{i=1}^{2n-1}\chi_i(t_i)$ with $Re(\chi_i)>0$, such that $|W(a)|$ is majorised by a sum of functions of the form $S(t_1,\dots,t_{2n-1})\delta_{B_{2n-1}}^{-1/2}(t)C_\chi(t)$, where S is a Schwartz function on F^{n-1}, and C_χ belongs to X. Hence we only need to consider the convergence of

$$\int_{A_{2n-1}} C_\chi(t(a))S(t(a))|a|^{s-1/2}d^*a=\int_{A_{2n-1}} C_\chi(t)S(t)\prod_{i=2}^{2n-2}|t_i|^{i-1}\prod_{i=1}^{2n-1}|t_i|^{i(s-1/2)}d^*t.$$

The statement follows, taking $\epsilon=min(-Re(\chi_j))$ for C_χ in X. □

Now we state our second result, about the nonvanishing of our integrals at $1/2$ for good choices of W and Φ. The proof of this proposition, as that of the previous one, will be an easy adaptation of the techniques of [J-S], though we followed even more closely the version of [F-Z].

Proposition 2.7. *Let s be a complex number with $Re(s)\geq 1/2-\epsilon$. There is W in $W(\pi,\theta)$, and Φ in $\mathcal{S}(F^n)$, such that $\Psi(s,W,\Phi)$ is nonzero.*

Proof. If not, $\Psi(s,W,\Phi)$ is zero for every W in $W(\pi,\theta)$, and Φ in $S(F^n)$. We are first going to prove that this implies that

$$\int_{N_{2n-1}^\sigma\backslash H_{n,n-1}} W\begin{pmatrix} h & \\ & 1\end{pmatrix}|h|^{s-1/2}dh=0$$

for every W in $W(\pi,\theta)$. Indeed, one has

$$\int_{N_{2n}^\sigma\backslash G_{2n}^\sigma} W(h)\Phi(L_n(h_2))|h|^s\delta(h)^{-1/2}dh=\int_{N_{2n}\backslash G_{2n}^\sigma} W(g)\Phi(L_n(g))|h|^{s-1/2}|h_2|dh$$

$$=\int_{P_{2n}^\sigma\backslash G_{2n}^\sigma}\left(\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma} W(ph)|ph|^{s-1/2}dp\right)\Phi(L_n(h_2))d\bar{h}$$

where $|h_2|dh$ is quasi-invariant on $N_{2n}^\sigma\backslash G_{2n}^\sigma$, and $d\bar{h}$ is quasi-invariant on $P_{2n}^\sigma\backslash G_{2n}^\sigma$.

But $P_{2n}^\sigma\backslash G_{2n}^\sigma \simeq F^n-\{0\}$ via $\bar{h}\mapsto L_n(h_2)$, and the Lebesgues measure on $F^n-\{0\}$ corresponds to $d\bar{h}$ via this homeomorphism. Hence, denoting $F(\bar{h})=\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma} W(ph)|ph|^{s-1/2}dp$, one has that for every Φ:

$$\int_{F^n-\{0\}} F(x)\Phi(x)dx=0,$$

in particular $F(0,\dots,0,1)=0$ (taking Φ approximating the Dirac measure supported at $(0,\dots,0,1)$), hence $\int_{N_{2n}^\sigma\backslash P_{2n}^\sigma} W(p)|p|^{s-1/2}dp=0$.
Then, one checks (see section 2 of [J-S]), that for every $\Phi\in\mathcal{S}(F^{n-1})$, the map $W_\phi: g\mapsto \int_{F^{n-1}} W(gu^\sigma(x))\Phi(x)dx$, where u^σ is the natural isomorphism between F^{n-1} and U_{2n}^σ, belongs to $W(\pi,\theta)$ again. But $W_\phi\begin{pmatrix} h & \\ & 1\end{pmatrix}=W\begin{pmatrix} h & \\ & 1\end{pmatrix}\widehat{\Phi}(L_n(h_1))$, hence

$$\int_{N_{2n-1}^\sigma\backslash H_{n,n-1}} W\begin{pmatrix} h & \\ & 1\end{pmatrix}\widehat{\Phi}(L_n(h_1))|h|^{s-1/2}dh$$

is zero for every W and Φ, which in turn implies that $\int_{N_{2n-2}^\sigma\backslash H_{n-1,n-1}} W\begin{pmatrix} h & \\ & I_2\end{pmatrix}|h|^{s-1/2}dh=0$ for every W. Continuing the process, we obtain $W(I_{2n})=0$ for every W, a contradiction. We didn't check the convergence of our integrals at each step, but it follows from Fubini's theorem. □

3 The global theory

3.1 The Eisenstein series

In the global case, let π be a smooth cuspidal representation of $G(A)$ with trivial central character, ϕ a cusp form in the space of π, and Φ an element of the Schwartz space $\mathcal{S}(A^n)$. We denote by χ a character of $H_{n,n}(A)$ of the form $h(h_1,h_2)\mapsto \alpha(det(h_1)/det(h_2)$ for α a character of A^*/k^*, and by δ again the character $h(h_1,h_2)\mapsto |h_1|/|h_2|$ of $H_{n,n}(A)$. Then we define

$$f_{\chi,\Phi}(s,h)=|h|^s\chi(h)\delta^{-1/2}(h)\int_{A^*}\Phi(aL_n(h_2))|a|^{2ns}d^*a$$

for h in H, s in the half plane $Re(s)>1/2n$, where the integral converges absolutely. It is obvious that $f_{\chi,\Phi}(s,h)$ is $Z_{2n}(k)P_{2n}^\sigma(k)$-invariant on the left.
Now we average on f on $P_{2n}^\sigma(A)$ to obtain the following Eisenstein series:

$$E(s,h,\chi,\Phi)=\sum_{\gamma\in Z_{2n}(k)P_{2n}^\sigma(k)\backslash H_{n,n}(k)} f_{\chi,\Phi}(s,\gamma h).$$

One can rewrite $E(s,h,\chi,\Phi)$ as

$$|h|^s\chi(h)\delta^{-1/2}(h)\int_{k^*\backslash A^*}\Theta'_\Phi(a,h)|a|^{2ns}d^*a,$$

where $\Theta'_\Phi(a,h)=\sum_{\xi\in k^n-\{0\}}\Phi(a\xi L_n(h_2))$.
According to Lemmas 11.5 and 11.6 of [G-J], it is absolutely convergent for $Re(s)>1/2$ large enough, uniformly on compact subsets of $H_{n,n}(k)\backslash H_{n,n}(A)$, and of moderate growth with respect to g.

Write $\Theta_\Phi(a,h)$ for $\Theta'_\Phi(a,h)+\Phi(0)$, then the Poisson formula for Θ_Φ gives:

$$\Theta_\Phi(a,h)=|a|^{-n}|h_2|^{-1}\Theta_{\hat{\Phi}}(a^{-1},{}^th^{-1}).$$

This allows us to write

$$E(s,h,\chi,\Phi)=|h|^s\chi(h)\delta^{-1/2}(h)\int_{|a|\geq 1}\Theta'_\Phi(a,h)|a|^{2ns}d^*a+|h|^{s-1/2}\chi(h)\int_{|a|\geq 1}\Theta'_{\widehat{\Phi}}(a,{}^th^{-1})|a|^{n(1-2s)}d^*a+u(s)$$

with $u(s)=-c\Phi(0)|h|^s\chi(h)\delta^{-1/2}(h)/2s+c\widehat{\Phi}(0)\chi(h)|h|^{s-1/2}/(1-2s)$.

We deduce from this the following proposition:

Proposition 3.1. *$E(s,h,\chi,\Phi)$ admits a meromorphic extension to $\mathbb{C}$, has at most simple poles at 0 and 1/2, and satisfies the functional equation:*

$$E(1/2-s,{}^th^{-1},\chi^{-1}\delta^{1/2},\widehat{\Phi})=E(s,h,\chi,\Phi).$$

Then the following integral converges absolutely for $Re(s)>1/2$:

$$I(s,\phi,\chi,\Phi)=\int_{Z_{2n}(A)H_{n,n}(k)\backslash H_{n,n}(A)}E(s,h,\chi,\Phi)\phi(h)dh.$$

When χ is trivial, we simply omit it in the notations.

Theorem 3.1. *The integral $I(s,\phi,\chi,\Phi)$ extends meromorphically to $\mathbb{C}$, with poles at most simple at 0 and 1/2, moreover, a pole at 1/2 occurs if and only if the global χ^{-1}-period*

$$\int_{Z_{2n}(A)H_{n,n}(k)\backslash H_{n,n}(A)}\chi(h)\phi(h)dh$$

is not zero, and $\widehat{\Phi}(0)\neq 0$. The integral $I(s,\phi,\chi,\Phi)$ also admits the following functional equation:

$$I(1/2-s,\tilde{\phi},\chi^{-1}\delta^{1/2},\widehat{\Phi})=I(s,\phi,\chi,\Phi),$$

where $\tilde{\phi}:g\mapsto\phi({}^tg^{-1})$.

Proof. It is clear that the residue of $I(s,\phi,\chi,\Phi)$ at 1/2 is

$$c\widehat{\Phi}(0)\int_{Z_{2n}(A)H_{n,n}(k)\backslash H_{n,n}(A)}\chi(h)\phi(h)dh,$$

hence the result about periods.

□

3.2 The Euler factorisation

Let θ be a nontrivial character of A/k, we denote by W_ϕ the Whittaker function on $G_{2n}(A)$ associated to ϕ, and we let

$$\Psi(s,W_\phi,\chi,\Phi)=\int_{N^\sigma_{2n}(A)\backslash H_{n,n}(A)}W(h)\Phi(\eta h)\chi(h)|h|^s\delta^{-1/2}(h)dh,$$

this integral converges absolutely for $Re(s)$ large by classical gauge estimates of section 13 of [J-P-S], and is the product of the similar local integrals. We will need the following expansion of cusp forms on the mirabolic subgroup, which can be found in [C], p.5.

Proposition 3.2. *Let ϕ be a cusp form on $P_l(A)$, then for any p in $P_l(A)$, then ϕ has the partial Fourier expansion, with uniform convergence for p in compact subsets of $P_l(A)$:*

$$\phi(p)=\sum_{\gamma\in P_{l-1}(k)\backslash G_{l-1}(k)}\left(\int_{y\in(A/k)^{l-1}}\phi(u(y)\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}p)\theta^{-1}(y_{n-1})dy\right)$$

Theorem 3.2. *One has the identity $I(s,\phi,\chi,\Phi)=\Psi(s,W_\phi,\chi,\Phi)$ for $Re(s)$ large.*

Proof. Denoting by χ_s the character $\chi\delta^{-1/2}|.|^s$ of $H_{n,n}(A)$, we start with

$$I(s,\phi,\chi,\Phi)=\int_{Z_{2n}(A)H_{n,n}(k)\backslash H_{n,n}(A)} E(s,h,\chi,\Phi)\phi(h)dh=\int_{Z_{2n}(A)P_{2n}^\sigma(k)\backslash H_{n,n}(A)} f_{\chi,\Phi}(s,h)\phi(h)dh$$

$$=\int_{P_{2n}^\sigma(k)\backslash H_{n,n}(A)} \Phi(Ln(h_2))\phi(h)\chi_s(h)dh$$

We denote for the moment $\Phi(Ln(h_2))\chi_s(h)$ by $F(h)$, and for l between 1 and $n-1$, we write:

$$I_l=\int_{P_{2l}^\sigma(k)(U_{2l+1}\ldots U_{2n})^\sigma(A)\backslash H_{n,n}(A)} F(h)\left(\int_{(U_{2l+1}\ldots U_{2n})(k)\backslash(U_{2l+1}\ldots U_{2n})(A)}\phi(nh)\theta^{-1}(n)dn\right)dh$$

and we also write $I_0=\Psi(W_\phi,\Phi,s)$ and $I_n=I(s,\phi,\chi,\Phi)$.

To prove the theorem, we only need to prove that $I_l=I_{l+1}$, that's what we do now. According to Proposition 3.2, applied to $\phi_1(h)=\int_{(U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+2}\ldots U_{2n})(A)}\phi(nh)\theta^{-1}(n)dn$ restricted to $w'_{2l+1}(P_{l+1})$, one has

$$\sum_{\gamma\in P_l(k)\backslash G_l(k)}\left(\int_{(U_{2l+1}U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+1}U_{2l+2}\ldots U_{2n})(A)}\phi(nw'_{2l+1}(\gamma)h)\theta^{-1}(n)dn\right)$$

$$=\int_{n\in(U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+2}\ldots U_{2n})(A)}\left(\int_{u\in U_{2l+1}^\sigma(k)\backslash U_{2l+1}^\sigma(A)}\phi(nuh)du\right)\theta^{-1}(n)dn$$

But as a system of representatives of $P_{2l}^\sigma(k)\backslash P_{2l+1}^\sigma(k)$ is given by the products $\gamma_0\gamma_1$, for γ_0 in $P_{2l}^\sigma(k)\cap w'_{2l+1}(G_l)(k)\backslash w'_{2l+1}(G_l)=w'_{2l+1}(P_l)(k)\backslash w'_{2l+1}(G_l)(k)$, and γ_1 in $P_{2l}^\sigma(k)w'_{2l+1}(G_l)(k)\backslash P_{2l+1}^\sigma(k)\simeq U_{2l+1}^\sigma(k)$, we get:

$$\sum_{\gamma\in P_{2l}^\sigma(k)\backslash P_{2l+1}^\sigma(k)}\left(\int_{(U_{2l+1}U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+1}U_{2l+2}\ldots U_{2n})(A)}\phi(nw'_{2l+1}(\gamma)h)\theta^{-1}(n)dn\right)$$

$$=\int_{n\in(U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+2}\ldots U_{2n})(A)}\left(\int_{u\in U_{2l+1}^\sigma(A)}\phi(nuh)du\right)\theta^{-1}(n)dn$$

Finally replacing in I_l, one obtains:

$$I_l=J_l=\int_{P_{2l+1}^\sigma(k)(U_{2l+2}\ldots U_{2n})^\sigma(A)\backslash H_{n,n}(A)} F(h)\left(\int_{(U_{2l+2}\ldots U_{2n})(k)\backslash(U_{2l+2}\ldots U_{2n})(A)}\phi(nh)\theta^{-1}(n)dn\right)dh$$

But applying again Proposition 3.2, this time to $\phi_2(h)=\int_{(U_{2l+3}\ldots U_{2n})(k)\backslash(U_{2l+3}\ldots U_{2n})(A)}\phi(nh)\theta^{-1}(n)dn$ restricted to $w'_{2l+2}(P_{l+2})$, one has

$$\sum_{\gamma\in P_{l+1}(k)\backslash G_{l+1}(k)}\left(\int_{(U_{2l+2}U_{2l+3}\ldots U_{2n})(k)\backslash(U_{2l+2}U_{2l+3}\ldots U_{2n})(A)}\phi(nw'_{2l+2}(\gamma)h)\theta^{-1}(n)dn\right)$$

$$= \int\limits_{n\in(U_{2l+3}\dots U_{2n})(k)\backslash(U_{2l+3}\dots U_{2n})(A)} \left(\int\limits_{u\in U^{\sigma}_{2l+2}(k)\backslash U^{\sigma}_{2l+2}(A)} \phi(nuh)du \right) \theta^{-1}(n)dn$$

But as a system of representatives of $P^{\sigma}_{2l+1}(k)\backslash P^{\sigma}_{2l+2}(k)$ is given by the products $\gamma_0\gamma_1$, for γ_0 in $P^{\sigma}_{2l+1}(k)\cap w'_{2l+2}(G_{l+1})(k)\backslash w'_{2l+2}(G_{l+1}) = w'_{2l+2}(P_{l+1})(k)\backslash w'_{2l+2}(G_{l+1})(k)$, and γ_1 in the set $P^{\sigma}_{2l+1}(k)w'_{2l+2}(G_{l+1})(k)\backslash P^{\sigma}_{2l+2}(k) \simeq U^{\sigma}_{2l+2}(k)$, we get:

$$\sum_{\gamma\in P^{\sigma}_{2l+1}(k)\backslash P^{\sigma}_{2l+2}(k)} \left(\int\limits_{(U_{2l+2}U_{2l+3}\dots U_{2n})(k)\backslash(U_{2l+2}U_{2l+3}\dots U_{2n})(A)} \phi(nw'_{2l+2}(\gamma)h)\theta^{-1}(n)dn \right)$$

$$= \int\limits_{n\in(U_{2l+3}\dots U_{2n})(k)\backslash(U_{2l+3}\dots U_{2n})(A)} \left(\int\limits_{u\in U^{\sigma}_{2l+2}(A)} \phi(nuh)du \right) \theta^{-1}(n)dn$$

Finally replacing in J_l, one obtains $J_l = I_{l+1}$, and this proves the theorem. □

As a corollary, we see that $\Psi(s, W_\phi, \chi, \Phi)$ extends to a meromorphic function (namely $I(s,\phi,\chi,\Phi)$). Writing π as the restricted tensor product $\otimes_\nu \pi_\nu$, for any $W = \prod_\nu W_\nu$ in $W(\pi,\theta)$, any decomposable $\Phi = \prod_\nu \Phi_\nu$ in $\mathcal{S}(A^n)$, one has

$$\Psi(s, W, \chi, \Phi) = \prod_\nu \Psi(s, W_\nu, \chi_\nu, \Phi_\nu).$$

3.3 The partial L-function

Let $\pi = \otimes_\nu \pi_\nu$ be a cuspidal automorphic representation of $G_{2n}(A)$, and S the finite set of places of k, such that π_ν is archimedean or ramified. We define the partial L-function $L^{lin,S}(s,\pi)$ to be the product $\prod_{\nu\notin S} L(s,\pi_\nu)L(2s,\Lambda^2,\pi_\nu)$, where $L(s,\pi_\nu)$ and $L(2s,\Lambda^2,\pi_\nu)$ are the corresponding L-functions of the Galois parameter of π_ν. Hence, if θ_ν has conductor $\mathfrak{O}$ at every unramified place ν, the function $L^{lin,S}(s,\pi)$ is the product $\prod_{\nu\notin S} \Psi(s, W^0_\nu, \Phi^0_\nu)$. Because of this, we see that it is meromorphic (it is equal to $\Psi(s,W,\Phi)/\prod_{\nu\in S}\Psi(s,W_\nu,\Phi_\nu)$ for a well chosen Φ an W). We now show that $L^{lin,S}(s,\pi)$ has at most a simple pole at $1/2$, and that this happens if and only if π admits a global period.

Theorem 3.3. *The partial L-function $L^{lin,S}(s,\pi)$ is holomorphic for $Re(s) > 1/2$ and has a pole at $1/2$ if and only if π has an $H_{n,n}(A)$-period. If it is the case, this pole is simple.*

Proof. Let S_∞ be the archimedean places of k, and S_f the set of finite places in S. First, for ν in S_f, as in the proof of Theorem 2.1, we take W_ν and Φ_ν such that $\Psi(s,W_\nu,\Phi_\nu) = 1$ for all s (and $\widehat{\Phi_\nu}(0) \neq 0$ because Φ_ν is positive). For any $s_0 \geq 1/2$, if ν belongs to S_∞, it is possible to take W_{ν,s_0} and Φ_{ν,s_0} such that $\Psi(s, W_{\nu,s_0}, \Phi_{\nu,s_0})$ is convergent for $s \geq 1/2-\epsilon$, and $\neq 0$ for $s = s_0$ according to Propositions 2.6 and 2.7. Writing

$$W_{s_0} = \prod_{\nu\in S_\infty} W_{\nu,s_0} \prod_{\nu\in S_f} W_\nu \prod_{\nu\notin S} W^0_\nu$$

and

$$\Phi_{s_0} = \prod_{\nu\in S_\infty} \Phi_{\nu,s_0} \prod_{\nu\in S_f} \Phi_\nu \prod_{\nu\notin S} \Phi^0_\nu,$$

the theorem follows from the equality $\prod_{\nu\in S}\Psi(s,W_{\nu,s_0},\Phi_{\nu,s_0})L^{lin,S}(s,\pi) = \Psi(W_{s_0},\Phi_{s_0},s)$, and Theorems 3.2 and 3.1. □

We get as a corollary, a well-known theorem of Friedberg and Jacquet ([FJ]).

Theorem 3.4. *The cuspidal automorphic representation π of $G_{2n}(A)$ admits a global period if and only if $L^S(s,\Lambda^2,\pi)$ has a pole at 1, and $L(1/2,\pi)\neq 0$.*

Proof. It is well know (see [G-J]) that $L(s,\pi)$ is entire, hence $L^S(s,\pi)$. Moreover $L(1/2,\pi)=0$ if and only if $L^S(1/2,\pi)=0$. Indeed $L^S(s,\pi)$ is an entire multiple of $L(s,\pi)$, hence one implication. Using the Rankin-Selberg convolution for $G_{2n}(A)\times A^*$, then for any W in $W(\pi,\theta)$, denoting $\int_{A^*} W(a,1,\dots,1)|a|^{(n-1)/2}d^*a$ by $\Psi(s,W)$, $\prod_{\nu\in S}\int_{k_\nu^*} W(t_\nu,1,\dots,1)|t_\nu|^{(n-1)/2}d^*t_\nu$ by $\Psi(s,W_S)$, and $\prod_{\nu\in S} L(s,\pi_\nu)$ by $L_S(s,\pi)$, one has $\Psi(s,W_S)L^S(s,\pi)=[\Psi(s,W_S)/L_S(s,\pi)]L(s,\pi)$. But there is $\epsilon>0$ such that $\Psi(s,W_S)$ converge for $Re(s)>1/2-\epsilon$ according to the estimates for the W_ν's retriction to A_{2n-1} given in Proposition 3 of [J-S2]. Hence if $L^S(1/2,\pi)=0$, then $[\Psi(1/2,W_S)/L_S(1/2,\pi)]L(1/2,\pi)=0$, but one can always choose W such that

$$[\Psi(1/2,W_S)/L_S(1/2,\pi)]\neq 0,$$

and $L_S(1/2,\pi)=0$. It is also proved in [J-S2] that the partial square-exterior L-function $L^S(s,\Lambda^2,\pi)$ can have a pole at 1 which is at most simple. Now the theorem follows from the equality $L^{lin,S}(s,\pi)=L^S(s,\pi)L^S(2s,\Lambda^2,\pi)$ and Theorem 3.3. □

4 The odd case

In this section, we just state the results for the odd case, which is totally similar to the even case.

In the local non archimedean case, for π a generic representation of G_{2n+1}, the integrals we consider are the following for W in $W(\pi,\theta)$, and Φ in $\mathcal{C}_c^\infty(F^n)$:

$$\Psi(s,W,\chi,\Phi)=\int_{N_{2n+1}\cap H_{n+1,n}\backslash H_{n+1,n}} W(h)\Phi(L_{n+1}(h_1))|h|^s\chi(h)dh.$$

We have:

Theorem 4.1. *There is a real number $r_{\pi,\chi}$, such that each integral $\Psi(s,W,\chi,\Phi)$ converges for $Re(s)>r_\pi$. Moreover, when W and Φ vary in $W(\pi,\theta)$ and $C_c^\infty(F^n)$ respectively, they span a fractional ideal of $\mathbb{C}[q^s,q^{-s}]$ in $\mathbb{C}(q^{-s})$, generated by an Euler factor which we denote $L^{lin}(s,\pi,\chi)$.*

The following functional equation is satisfied.

Theorem 4.2.

$$\Psi(1/2-s,\tilde{W},\chi^{-1}\delta'^{1/2},\widehat{\Phi}^\theta)/L^{lin}(1/2-s,\pi^\vee,\chi^{-1}\delta'^{1/2})=\epsilon^{lin}(s,\pi,\chi,\theta)\Psi(s,W,\chi,\Phi)/L^{lin}(s,\pi,\chi)$$

for $\epsilon^{lin}(s,\pi,\chi,\theta)$ a unit of $\mathbb{C}[q^s,q^{-s}]$, where $\tilde{W}: g\mapsto W(w^tg^{-1})$ with w is the permutation matrix corresponding to $i\mapsto 2n+1-i$.

For a unitary generic representation π, we have the following theorem:

Theorem 4.3. *Let π be a generic unitary representation of $G_{2n}(F)$, and χ a character of $H_{n+1,n}$ of the form $\alpha(det(h_1)/det(h_2))$, with $Re(\alpha)\geq 0$ (in particular χ can be $\mathbf{1}$) then π is χ-distinguished if and only if $L^{lin}(s,\pi,\chi^{-1})$ has an exceptional pole at 0.*

In this case, the cuspidal computation gives the same formula as in remark 2.1. However, according to [M4], a cuspidal representation of G_{2n+1} can never be $(H_{n+1,n},|.|^{-s_0})$-distinguished for any complex number s_0, hence we actually obtain that $L^{lin}(s,\pi)=1$ when π is a cuspidal representation of $H_{n+1,n}$.

Let π^0 be a generic unramified representation of G_{2n+1}. The unramified computation gives again, thanks to the relation $\delta_{B_{2n+1}}(a)=\delta_{B_{2n+1}}(a')\delta_{B_{2n+1}}(a'')$, with $a'=(a_1,a_3,\dots,a_{2n+1})$ and $a''=(a_2,a_4,\dots,a_{2n})$, the equality:

$$\Psi(s, W^0, \Phi^0) = \sum_{\lambda \in \Lambda^{++}} s_\lambda(q^{-s}z) = L(s, \pi^0)L(s, \Lambda^2, \pi^0)$$

In the archimedean case, for π a generic representation of G_{2n+1}, the integrals

$$\Psi(s, W, \Phi) = \int_{N_{2n+1} \cap H_{n+1,n} \backslash H_{n+1,n}} W(h)\Phi(L_{n+1}(h_1))|h|^s dh$$

converge again for $Re(s) \geq 1/2 - \epsilon$ for a positive number ϵ depending on π. For any such s, one can chose W and Φ such that they don't vanish.

In the global situation, for Φ in $\mathcal{S}(A^n)$, we define

$$f_{\chi,\Phi}(s, h) = |h|^s \chi(h) \int_{A^*} \Phi(aL_{n+1}(h_1))|a|^{(2n+1)s} d^*a$$

for h in $H_{n+1,n}$. Associated is the Eisenstein series

$$E(s, h, \chi, \Phi) = \sum_{\gamma \in Z_{2n+1}(k)P^{\sigma}_{2n+1}(k) \backslash H_{n+1,n}(k)} f_{\chi,\Phi}(s, \gamma h),$$

which converges absolutely for $Re(s) > 1/2$, extends meromorphically to $\mathbb{C}$, with possible poles simple, and located at 0 and $1/2$. It satsifies the functional equation

$$E(1/2 - s, {}^t h^{-1}, \chi^{-1}\delta'^{1/2}, \widehat{\Phi}) = E(s, h, \chi, \Phi).$$

Then if π is a cuspidal automorphic representation of $G_{2n+1}(A)$, and ϕ is a cusp form in the space of π, we define for $Re(s) > 1/2$ the integral

$$I(s, \phi, \chi, \Phi) = \int_{Z_{2n+1}(A)H_{n+1,n}(k) \backslash H_{n+1,n}(A)} E(s, h, \chi, \Phi)\phi(h)dh,$$

which satisfies the statement of the following theorem.

Theorem 4.4. *The integral $I(s, \phi, \chi, \Phi)$ extends to an entire function on $\mathbb{C}$. The integral $I(s, \phi, \chi, \Phi)$ also admits the following functional equation:*

$$I(1/2 - s, \tilde{\phi}, \chi^{-1}\delta^{1/2}, \widehat{\Phi}) = I(s, \phi, \chi, \Phi),$$

where $\tilde{\phi} : g \mapsto \phi({}^t g^{-1})$.

The proof is the same up to the following extra argument. It is clear that the residue of $I(s, \phi, \chi, \Phi)$ at $1/2$ is

$$c\widehat{\Phi}(0) \int_{Z_{2n+1}(A)H_{n+1,n}(k) \backslash H_{n+1,n}(A)} \chi(h)\phi(h)dh,$$

but these integrals are to known to vanish according to Proposition 2.1 of [B-F], hence there is actually no pole at $1/2$.

Again, we define

$$\Psi(s, W_\phi, \chi, \Phi) = \int_{N_{2n+1}(A) \cap H_{n+1,n}(A) \backslash H_{n+1,n}(A)} W_\phi(h)\Phi(L_{n+1}(h_1))|h|^s \chi(h)dh,$$

and this integral converge for $Re(s)$ large, and is in fact equal to $I(s, \phi, \chi, \Phi)$.

From this we deduce that the partial L-function $L^{lin}(s, \pi)$ is meromorphic, and holomorphic for $Re(s) \geq 1/2 - \epsilon$, for some positive ϵ (corresponding to the ϵ of the archimedean case).

5 Conclusion

It would be nice to know, in the non-archimedean case, that if π is an irreducible representation of $G_k(F)$, then $L^{lin}(s,\pi) = L(s,\phi(\pi))L(2s,\Lambda^2\phi(\pi))$ for $\phi(\pi)$ the Galois parameter of π.
A possible strategy is the following: prove the equality for the discrete series, and prove the multiplicativity relation $L^{lin}(s,\pi_1\times\pi_2) = L^{lin}(s,\pi_1)L^{lin}(s,\pi_1)L(s,\pi_1\times\pi_2)$ where $\times$ denotes the normalised parabolic induction.
For the first assertion, we think that it should be a consequence of the known relation between local and Shalika models for discrete series, and the exceptional poles of the relevant L-functions, as well as the known equality $L(s,\pi)L(2s,\Lambda^2(\pi)) = L(s,\phi(\pi))L(2s,\Lambda^2\phi(\pi))$ (see [K-R]). In the previous equality, the left side of the equality is the Godement-Jacquet L-function times the Jacquet-Shalika exterior-square L-function.
For the second assertion, we think it could follow from the technique of Cogdell and Piatetski-Shapiro used in [M2] for proving the inductivity relation of the Asai L-function, and the following theorem, that we state as a conjecture, which is analogous to the main theorem of [M3].

Conjecture 5.1. *Let $\pi = \Delta_1\times\cdots\times\Delta_r$ be a generic representation of $G_{2n}(F)$ (which is the commutative product the of discrete series Δ_i). Then π is $H_{n,n}$-distinguished if and only if one can write the set $\{\Delta_1,\dots,\Delta_r\}$ as a partition of pairs $\{\Delta,\Delta^\vee\}$, and singletons $\{\Delta\}$, with Δ a discrete series of $G_{2k}(F)$ with $k\leq n$ which admits a linear period.*
Let $\pi = \Delta_1\times\cdots\times\Delta_r$ be a generic representation of $G_{2n+1}(F)$. Then π admits a linear period if and only if one can order the set $\{\Delta_1,\dots,\Delta_r\}$ such that $\Delta_1 = 1$, and write $\{\Delta_2,\dots,\Delta_r\}$ as a partition of pairs $\{\Delta,\Delta^\vee\}$, and singletons $\{\Delta\}$, with Δ a discrete series of $G_{2k}(F)$ with $k\leq n$ which admits a linear period.

This would actually give a classification of generic representations admitting a linear period in terms of cuspidal representations, according to Theorem 6.1. of [M5].

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